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Dark Energy Model with Canonical Scalar Field and Non-Linear Born–Infeld Type Scalar Field

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In this paper, we propose the non-linear Born–Infeld scalar field and canonical scalar field dark energy models with the potential $\frac{\lambda}{4}(\phi^2 - \sigma^2)^2 + V_0$, which admits late time de Sitter attractor solution. The attractor solution corresponds to an equation of state $\omega_\phi \rightarrow -1$ and a cosmic density parameter $\Omega_\phi \rightarrow 1$, which are important features for a dark energy model that can meet the current observations. dark energy; canonical scalar field, non-linear Born–Infeld type scalar field, attractor solution.

KEY WORDS: dark energy; canonical scalar field; non-linear Born–Infeld type scalar field; attractor solution.
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1. INTRODUCTION

Measurements of the redshift-luminosity distance relation using high redshift type Ia supernovas combined with cosmic microwave background (CMB) and galaxy clusters data appear to suggest that the present Universe is flat and undergoing a period of accelerated expansion (Alverson *et al.*, 2002; Bennett *et al.*, 2003; Netterfield *et al.*, 2002; Perlmuter *et al.*, 1999; Riess *et al.*, 1998; Tonry *et al.*, 2003), with the energy density splits into two main contributions, $\Omega_{\text{matter}} \approx 1/3$ and $\Omega_\Lambda = 2/3$. The roughly two-thirds of the energy density in our Universe results from a kind of dark energy that has a negative pressure and can drive the accelerating expansion. Many candidates for dark energy have been proposed so far to fit the current observations. Among these models, the most typical ones are the cosmological constant and a time varying scalar field with positive or negative kinetic energy evolving in a specific potential, referred to them as "quintessence" (Caldwell *et al.*, 1998; Coble *et al.*, 1997; Padmanabhan, 2003; Peebles and Ratra, 2003; Ratra and Peebles, 1988) or "phantom" (Caldwell,

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2002; Carroll *et al.*, 2003; Hao and Li, 2003a; Singh *et al.*, 2003). Successful dark energy models also share some common features: (i) they should have an effective equation of state $\omega < -1/3$ (where $\omega = p/\rho$, $\ddot{a} \propto -(\rho + 3p)$) so as to accelerate the expansion of the universe at recent epoch; (ii) they should be negligible comparing with radiation and matter in the early epoch of the universe so as not to affect the primordial nucleosynthesis while dominating over the matter in a recent epoch; and (iii) they should not be very sensitive to initial conditions so as to alleviate fine-tuning problems. For the canonical scalar field model and non-linear Born–Infeld type scalar field model, a great deal of effort has been made to determine the appropriate potential $V(\phi)$ that could explain current cosmological observations (Barreiro *et al.*, 2000; Caldwell *et al.*, 1998; Zlatev *et al.*, 1999). In this paper, we propose the potential as:

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - \sigma^2)^2 + V_0 \tag{1}$$

which has a nonvanishing minimum. The sufficient condition for the existence of a viable cosmological model with a late time de Sitter attractor solution should be that: the potential of the field has a nonvanishing minimum (Hao and Li, 2003b). The dynamical system of the models admit late time de Sitter attractor solution, which meet all the above three points.

The paper is organized as follows: In Section 2, a sufficient condition for dark energy with a late time de Sitter attractor for the canonical scalar field model is discussed. In Section 3, we construct the non-linear Born–Infeld scalar field dark energy model and compare two models by numerical method. Section 4 is summary.

2. DARK ENERGY MODEL WITH A CANONICAL SCALAR FIELD

In this section, we will study the case that the dark energy is mimicked by a scalar field expressed by a canonical Lagrangian. We will work in the flat Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{2}$$

The equation of motion for the scalar field with a canonical Lagrangian is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\phi)}{\partial \phi} = 0 \tag{3}$$

In order to gain more insight into the dynamical system, we introduce the new dimensionless variables

$$x = \frac{\phi}{\phi_0} \qquad y = \frac{\dot{\phi}}{\phi_0^2} \qquad N = \ln a \tag{4}$$

Then the above equation can be rewritten as

$$\frac{dx}{dN} = \frac{\phi_0 y}{H}$$
$$\frac{dy}{dN} = -3y - \frac{V'(x)}{\phi_0^3 H} \tag{5}$$

where the prime denotes the derivative with respect to x and H is a Hubble parameter that could be rewritten as

$$H^2 = H_i^2 E^2(N) \tag{6}$$

where H_i denotes the Hubble parameter at an initial time. $\Omega_{M,i}$ and $\Omega_{r,i}$ are the cosmic density parameters for matter and radiation at the initial time. We also choose the initial scale factor $a_i = 1$ and $E(N)$ is defined as

$$E(N) = \left[\frac{k}{3H_i^2}\left(\frac{\phi_0^4 y^2}{2} + V(x)\right) + \Omega_{M,i}e^{-3N} + \Omega_{r,i}e^{-4N}\right]^{1/2} \tag{7}$$

where $k = 8\pi G$. At late time, N goes to be very large and the contribution from matter and radiation in Eq. (7) becomes negligible comparing with the scalar field. To see this more clearly, we take the example that when the equation of state of the dark energy is constant ω_ϕ must be less than $-1/3$ so as to accelerate the expansion of the Universe. Thus the dark energy component will evolve with N as $\Omega_\phi e^{-3(1+\omega_\phi)N}$, which dissipates slower than matter and radiation. So, at late time, we will have

$$\frac{dx}{dN} = \sqrt{\frac{3}{k}}\frac{\phi_0 y}{\sqrt{\phi_0^4 y^2/2 + V(x)}}$$
$$\frac{dy}{dN} = -3y - \sqrt{\frac{3}{k}}\frac{V'(x)}{\phi_0^3\sqrt{\phi_0^4 y^2/2 + V(x)}} \tag{8}$$

The critical point of the above autonomous system is $(x_c, 0)$, where x_c is defined by $V'(x_c) = 0$. Linearizing the Eq. (8) about the critical point, we will have

$$\frac{dx}{dN} = \sqrt{\frac{3}{kV(x_c)}}\phi_0 y$$
$$\frac{dy}{dN} = -3y - \sqrt{\frac{3}{kV(x_c)}}\frac{V''(x_c)x}{\phi_0^3} \tag{9}$$

The type of the critical point is determined by the eigenequation of system

$$\lambda^2 + \alpha\lambda + \beta = 0 \tag{10}$$

where $\alpha = 3$ and $\beta = [3V''(x_c)]/[kV(x_c)\phi_0^2]$, the two eigenvalues are $\lambda_{1,2} = (-\alpha \pm \sqrt{\alpha^2 - 4\beta})/2$, For a positive potential $V(x)$, if $V''(x_c) > 0$, then the critical

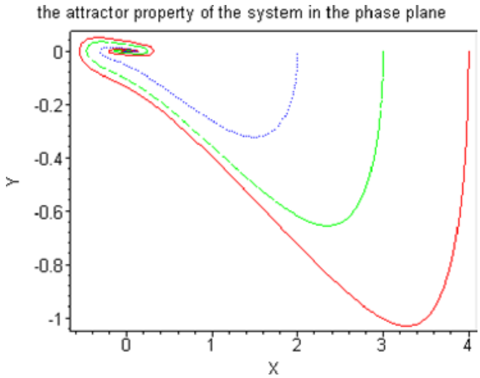

Fig. 1. The attractor property of the system in the phase plane. We can easily find that the system admits a attractor solution. We choose $x_i = 2, y_i = 0$ (*dotted line*); $x_i = 3, y_i = 0$ (*dashed line*); $x_i = 4, y_i = 0$ (*solid line*).

point is a stable node. This is to say that the dynamical system has a stable critical point at the minimum of the potential. This critical point corresponds to a late time attractor solution. These properties have been further confirmed by the numerical analysis (Fig. 1). Next, let us read out the physical implications when the system is at the attractor regime. The cosmic density parameter for the dark energy is

$$\Omega_\phi = \frac{k\left[\phi_0^4 y^2/2 + V(x)\right]}{3H_i^2 E^2(N)} \tag{11}$$

and the equation of state of the scalar field is

$$\omega_\phi = \frac{\phi_0^4 y^2 - 2V(x)}{\phi_0^4 y^2 + 2V(x)} \tag{12}$$

Clearly, from Eqs. (11) and (12), one can easily find that $\omega_\phi = -1$ and $\Omega_\phi = 1$ when the scalar field is dominant over the matter and radiation in the universe.

We choose a widely studied potential as Eq. (1). Substituting this potential into Eq. (4), we obtain

$$\begin{aligned}\frac{dx}{dN} &= \frac{\phi_0 y}{H_i\left[\frac{k}{3H_i^2}\left(\frac{\phi_0^4 y^2}{2} + \frac{\lambda}{4}\left(x^2\phi_0^2 - \sigma^2\right)^2 + V_0\right) + \Omega_{M,i}e^{-3N} + \Omega_{r,i}e^{-4N}\right]^{1/2}} \\ \frac{dy}{dN} &= -3y \\ &\quad - \frac{\lambda\phi_0\left(x^2\phi_0^2 - \sigma^2\right)x}{\phi_0^3 H_i\left[\frac{k}{3H_i^2}\left(\frac{\phi_0^4 y^2}{2} + \frac{\lambda}{4}\left(x^2\phi_0^2 - \sigma^2\right)^2 + V_0\right) + \Omega_{M,i}e^{-3N} + \Omega_{r,i}e^{-4N}\right]^{1/2}}\end{aligned} \tag{13}$$

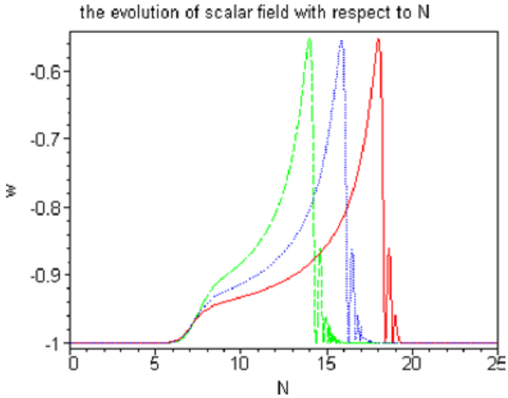

Fig. 2. The evolution of ω with respect to N. The equation of state ω moves towards zero.

We choose $k\phi_0^4/3H_i^2 = 3 \times 10^{-2}$, $\sigma_0^2/H_i\phi_0 = 10^{-5}$ and $\lambda = 10^{-1}$. We specify starting point at the equipartition epoch ($\Omega_M = \Omega_r = 0.5$). The numerical results with different initial conditions are plotted in Figs. 1–3.

After this stage, ω fast drops and begins to oscillate. The equation of state is sufficiently negative and finally settles at -1. We choose same initial conditions with Fig. 1.

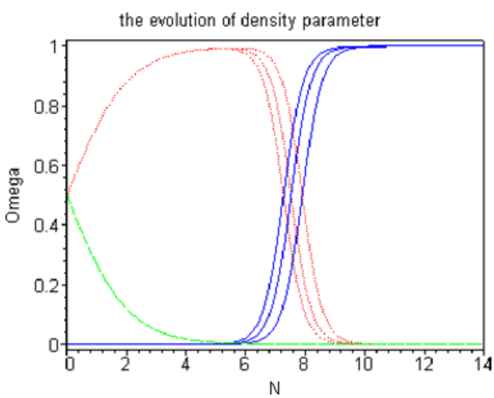

Fig. 3. The evolution of density parameter Ω with respect to N. *Solid line* is for scalar field, *dotted line* for matter and *dashed line* for radiation. Since the parameters in the potential will not affect the shape of the plot significantly, we here plot for different initial x_i and y_i, which are same with the Fig. 1. The curves show that the canonical scalar field model has a wide range of initial conditions, which alleviates the fine-tuning problems.

3. DARK ENERGY MODEL WITH A NON-LINEAR BORN–INFELD TYPE SCALAR FIELD

Non-linear Born–Infeld theory has been considered widely in string theory and cosmology. In 1934 (Born and Infeld, 1934), Born and Infeld put forward a theory of non-linear electromagnetic field to resolve the singularity in classical electromagnetic dynamics. The lagrangian density is

$$L_{BI} = b^2\left[1 - \sqrt{1 + \left(\frac{1}{2b^2}\right) F_{\mu\gamma}F^{\mu\gamma}}\,\right] \tag{14}$$

In order to describe the process of meson multiple production connected with strong field regime (Heisenberg, 1952, 1946), Heisenberg proposed the following nonlinear scalar field lagrangian firstly

$$L = \frac{1}{\eta}\left[1 - \sqrt{1 - \eta g^{\mu\gamma}\phi_{,\mu}\phi_{,\gamma}}\,\right] \tag{15}$$

This lagrangian density (Eq. (15)) possesses some interesting characteristics: (i) it is exceptional in the sense that shock waves do not develop under smooth or continuous (Taniuti, 1958), (ii) because nonlinearities have been introduced, nonsingular scalar field solutions can be generated, (iii) if $g^{\mu\gamma}\phi_{,\mu}\phi_{,\gamma} \ll \frac{1}{\eta}$, by Taylor expansion, Eq. (15) approximates to the lagrangian of linear scalar field,

$$\lim_{\eta\to 0} L = \frac{1}{2} g^{\mu\gamma}\phi_{,\mu}\phi_{,\gamma} \tag{16}$$

the linear theory is recovered. H.P. de Oliveira has investigated qualitatively the static and spherically symmetric solutions of this nonlinear scalar field (De Oliveira, 1995). Especially, if the potential $V(\phi)$ equals to $1/\eta$, we find that $p\rho = -1/\eta^2$, which describes a Chaplygin gas.

We have proposed a dark energy model based on the lagrangian Eq. (15) in literatures (Lu, 2005; Fang, 2005). We find that the universe of Born–Infeld scalar field with a potential can undergo a phase of accelerating expansion. We have the following lagrangian:

$$L = \frac{1}{\eta}\left[1 - \sqrt{1 - \eta\dot{\phi}^2}\,\right] - V(\phi) \tag{17}$$

The equation of motion for a scalar field with a Born–Infeld type Lagrangian is

$$\ddot{\phi} + 3H\dot{\phi}(1 - \eta\dot{\phi}^2) + V'(\phi)(1 - \eta\dot{\phi}^2)^{3/2} = 0 \tag{18}$$

Where the overdot represents the differentiation with respect to t and the prime denotes the differentiation with respect to ϕ. The Energy-moment tensor is

$$T^{\mu}_{\gamma} = -\frac{g^{\mu\rho}\phi_{,\gamma}\phi_{,\rho}}{\sqrt{1-\eta g^{\mu\gamma}\phi_{,\mu}\phi_{,\gamma}}} - \delta^{\mu}_{\gamma}L \tag{19}$$

From Eq. (19), we have

$$\rho_{\phi} = T^{0}_{0} = \frac{1}{\eta\sqrt{1-\eta\dot{\phi}^2}} - \frac{1}{\eta} + V(\phi) \tag{20}$$

$$p_{\phi} = -T^{i}_{i} = \frac{1}{\eta} - \frac{\sqrt{1-\eta\dot{\phi}^2}}{\eta} - V(\phi) \tag{21}$$

The equation of state is

$$\omega_{\phi} = \frac{p_{\phi}}{\rho_{\phi}} = -1 + \frac{\eta\dot{\phi}^2}{1+(\eta V(\phi)-1)\sqrt{1-\eta\dot{\phi}^2}} \tag{22}$$

By introducing the new dimensionless variables (note that the field in Born–Infeld Lagrangian has a different dimension from that in canonical Lagrangian)

$$x = \phi \qquad y = \dot{\phi} \qquad N = \ln a \tag{23}$$

The equation of motion could be reduced to

$$\begin{aligned} \frac{dx}{dN} &= \frac{y}{H} \\ \frac{dy}{dN} &= -3y(1-\eta y^2) - \frac{(1-\eta y^2)^{3/2}V'(x)}{H} \end{aligned} \tag{24}$$

Where H is the same as that defined by Eq. (6)

$$H^2 = H_i^2 E^2(N)$$

but with a different:

$$E(N) = \left[\frac{k}{3H_i^2}\left(V(x) - \frac{1}{\eta} + \frac{1}{\eta\sqrt{1-\eta y^2}}\right) + \Omega_{M,i}e^{-3N} + \Omega_{r,i}e^{-4N}\right]^{1/2} \tag{25}$$

Substituting the potential Eq. (1) into the Eqs. (24) and (25), we obtain:

$$\begin{aligned}
\frac{dx}{dN} &= \frac{y}{H_i} E^{-1}(N) \\
\frac{dy}{dN} &= -3y(1-\eta y^2) - \frac{(1-\eta y^2)^{3/2} \lambda \phi_0 \left(x^2 \phi_0^2 - \sigma^2\right) x}{H_i} E^{-1}(N) \\
E(N) &= \left[\frac{k}{3H_i^2} \left(\lambda/4 (\phi^2 - \sigma^2)^2 + V_0 - \frac{1}{\eta} + \frac{1}{\eta \sqrt{1-\eta y^2}} \right) \right. \\
&\qquad \left. + \Omega_{M,i} e^{-3N} + \Omega_{r,i} e^{-4N} \right]^{1/2}
\end{aligned} \tag{26}$$

where $k = 8\pi G$. In a similar fashion as in the previous section, we conclude that the contribution from matter and radiation to the Hubble parameter become negligible at late time. With the critical point $(x_c, 0)$ and $V'(x_c) = 0$, one can observe that the linearized autonomous system for the canonical scalar field is quite similar to that of the Born–Infeld type scalar field. Thus we can conclude that for positive potentials, which has a nonvanishing minimum , the Born–Infeld type scalar model has a stable critical point, which corresponds to the de Sitter attractor solution of the system.

We specify our starting point at the equipartition epoch, at which $\Omega_M = \Omega_r = 0.5$ and take $\sigma_0^2 / H_i \phi_0 = 10^{-6}$, $\lambda = 1$, $k = 1$. They are plotted in different initial conditions.

As we know, when $\eta \to 0$, the Non-linear Born–Infeld type scalar field model comes back to canonical scalar field model. In order to see the nonlinear effect, we plot the two models phase portraits in Figs. 8 and 9.

From the Figs. 8 and 9, we can see that the Non-linear Born–Infeld type scalar field model has a same evolution with the canonical scalar field model at the last, but the canonical scalar field has a obvious oscillation.

4. SUMMARY

So far, we have present the non-linear Born–Infeld scalar field and canonical scalar field dark energy models with the potential $\frac{\lambda}{4}(\phi^2 - \sigma^2)^2 + V_0$. From the Figs. 1–3 and 6–9 we conclude that they admit a late time de Sitter attractor and the attractor solution correspond to an equation of state $\omega_\phi \to -1$ and a cosmic density parameter $\Omega_\phi \to 1$, which are important features for a dark energy model. It can meet the current observations. From the Figs. 3 and 9 we can easily find that at earlier epoch of universe the scalar field (ρ_ϕ) can be negligible compared with radiation and matter, but in a very recent epoch the scalar field (ρ_ϕ) surpasses the matter and radiation density and becomes the dominant component. With the evolution of time, the energy density of ϕ field slows to a crawl and $\omega_\phi \to -1$ as $\Omega_\phi \to 1$ and the universe is driven into an accelerating phase. It

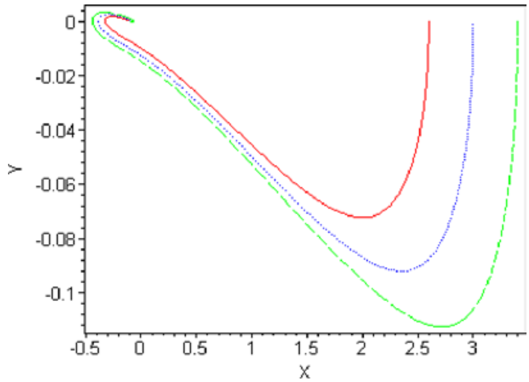

Fig. 4. The evolution of scalar field with respect to N. *Solid lines* is canonical scalar field model, *dotted line* and *dashed line* for Non-linear Born–Infeld type scalar field model, *dotted line* for $\eta = 0.1$, *dashed line* for $\eta = 0.05$.

is necessary to point out that the current observation date indicate that the cosmic density parameter of the dark energy is about $\Omega_\phi = 2/3$ and the equation of state is less than -0.82, therefore in the models analyzed in this paper, the current universe is just on the way to the attractor. We plot the numerical results

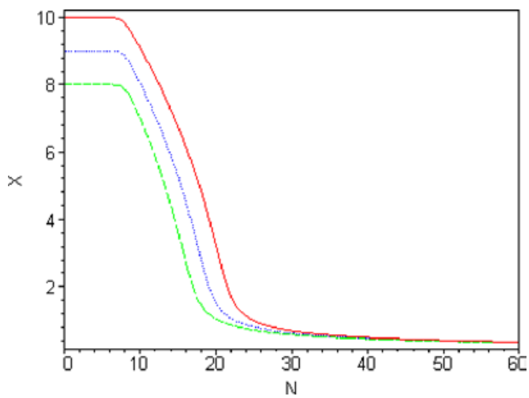

Fig. 5. The evolution of the equation of state ω of the scalar field with respect to N. *Solid lines* is canonical scalar field model, *dotted line* and *dashed line* for Non-linear Born–Infeld type scalar field model, *dotted line* for $\eta = 0.1$, *dashed line* for $\eta = 0.05$.

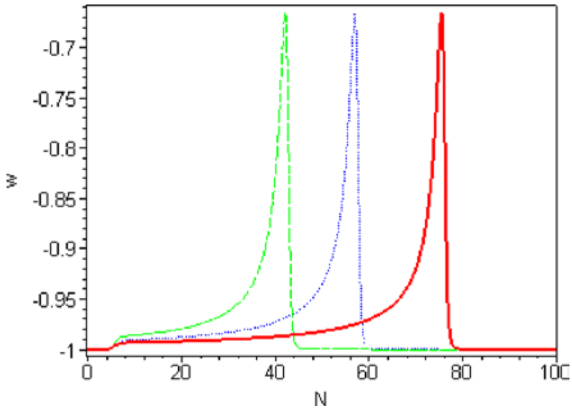

Fig. 6. Phase portrait of the model described by Eq. (26). Trajectories starting anywhere in the phase space end up at the stable critical point. We choose $x_i = 2.6$, $y_i = 0$ (*solid line*); $x_i = 3.0$, $y_i = 0$ (*dotted line*); $x_i = 3.4$, $y_i = 0$ (*dashed line*).

with different initial conditions. The model will always tend to be stable. So we can find the range of initial conditions is wide and it alleviates the fine tuning problems. The difference of the two models is that: the canonical scalar field has a obvious oscillation in the evolution of the scalar field and the equation of the state ω.

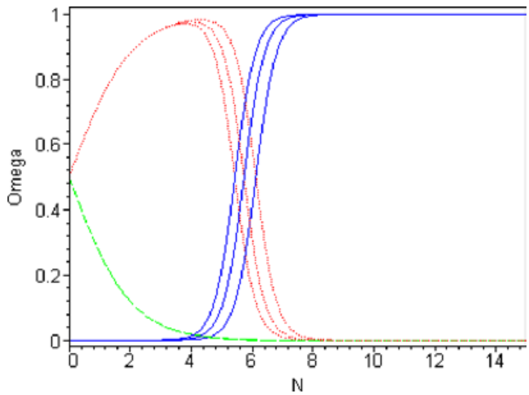

Fig. 7. The evolution of the scalar field model by Eq. (26) with respect to N in the presence of matter and radiation. We choose $x_i = 10$, $y_i = 0$ (*solid line*); $x_i = 9$, $y_i = 0$ (*dotted line*); $x_i = 8$, $y_i = 0$ (*dashed line*).

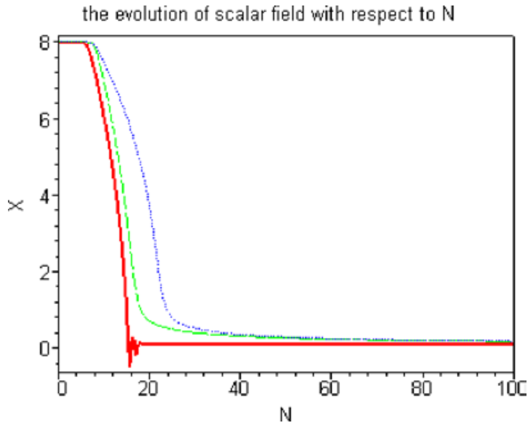

Fig. 8. The evolution of the equation of state ω of the scalar field described by Eq. (26). The equation of state ω also starts from nearly -1, then quickly evolves to the regime of lager than -1 but smaller than zero, turns back to execute the damped oscillation, and reaches to -1 for ever, just like the canonical scalar field dark energy model. We choose same initial conditions with Fig. 6.

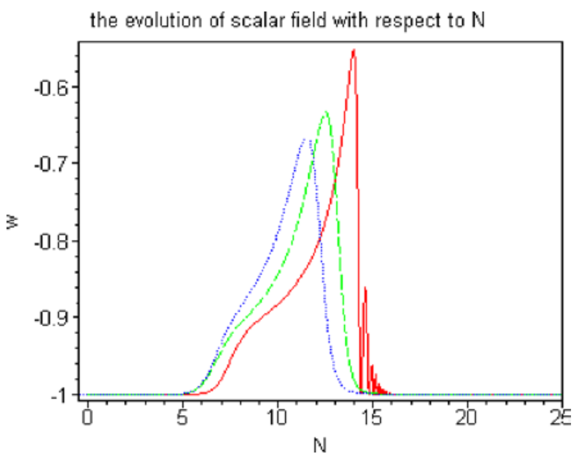

Fig. 9. The evolution of density parameter Ω with respect to N. The *solid line* is for scalar field, *dotted line* for matter and *dashed line* for radiation. Since the parameters in the potential will not affect the shape of the plot significantly, we here plot for different initial x_i and y_i, which are same with the Fig. 6.

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